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THE APPARENT WIDTH OF THE PLATE IN COMPRESSION

By Karl Marguerre

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THE APPARENT WIDTH OF THE PLATE IN COMPRESSION

By Karl Marguerre

The present report is a very simplifying derivation for the results of an investigation entitled: "The Load Capacity of a Plate Strip Stressed in Compression beyond the Buckling Limit" and published in 1937 in the Zeitschrift für angewandte Mathematik und Mechanik.

Following the discussion of the methods and results of other authors, the writer suggests an extension which is very desirable from the point of view of airplane design problems. It affords a practical theoretically evolved formula for the apparent width under an appreciably exceeded buckling load.

SUMMARY

The present extension of the customary stability investigation to include the supercritical range, proceeds in two steps. The first step considers the buckling form $w = f \cos \frac{\pi x}{l} \cos \frac{\pi y}{b}$ known from elementary theory, preserves the higher terms in f and yields, with the aid of the principle of virtual displacements, a relation which gives the decrease of the apparent strain stiffness at the instant of buckling (analytically expressed "the tangent to the new stress-strain curve above the critical load" (equation 5.5)).

The second step evolves on the basis of a formula containing several arbitrary values, from which the probably produced buckling form, with a greatly exceeded critical point, can be computed, and which affords a stress-strain curve (fig. 2) which reproduces with sufficient agreement the actual conditions existing in a zone

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*"Die mittragende Breite der gedrückten Platte." Luftfahrtforschung, vol. 14, no. 3, March 20, 1937, pp. 121-128.

$\epsilon^* < \epsilon_1 < 20\,\epsilon^*$ (as proved by comparison with experimental results). The apparent width is conveniently calculable (fig. 3) within the cited range with the aid of formula (7.8) or (7.9), which approximately comprise the result of the theory.

I. INTRODUCTION

The following investigation treats the load capacity of a rectangular plate stressed in compression in one direction (x) beyond the buckling limit. The plate is rotatably (i.e., free from moments) supported at all four sides by bending-resistant beams.

Before buckling, the axial compression $\bar{p}_x = -\bar{\sigma}_x$ is uniformly distributed and proportionality exists between the crushing $\epsilon_1 = -\epsilon$ and the compression $\bar{p}_x$ (according to the law of elasticity). Above the critical value ϵ^* of the crushing, the strip buckles - more in the middle than near the restrained sides - resulting in nonuniformly distributed axial compression; the centroidal axes, as commonly expressed, "do no longer fully contribute." The sought-for factor is the condition of form change in the buckled sheet and in particular, the new stress-strain curve; i.e., the relation between the mean value:

$$\bar{\bar{p}}_x = \frac{1}{b}\int_{-b/2}^{b/2} \bar{p}_x \, dy = p_1$$

of the compression and the mean crushing ϵ_1 (the crushing of the longitudinals):

$$p_1 = p_1(\epsilon_1) \quad \text{for} \quad \epsilon_1 > \epsilon^*$$

The crushing ϵ_1 (respectively, the amount of the pushing together of the transverse beams $\epsilon_1 l$) is chosen as the first independent variable of our problem; while as second independent variable, the lateral displacement (pushing together) $\epsilon_2 b$ of the longitudinal sides or else the mean value of the compression in transverse direction is introduced.

II. THE FUNDAMENTAL EQUATIONS

The hypotheses of classical plate theory (preservation of the normals, etc.) allow us to express the total stress-form change condition of the thin plate as function of the three displacements $u = u\ (x,y)$, $v = v\ (x,y)$, $w = w\ (x,y)$ of the plate middle. The problem of plate stretch (displacements u,v) may be reduced to the bipotential equation $\Delta\,\Delta\,\Phi = 0$ by having recourse to a "stress function" Φ; the deflection w follows the "plate equation" - i.e., the bipotential equation with interference term. The equations for Φ and w are unrelated.

Premise of this theory is the fundamental assumption of the "linearized" elasticity theory: that all displacements relative to the dimensions of the body, particularly as regards the plate thickness s, are small.

But a thin-walled sheet may undergo elastic deflections amounting to multiples of its plate thickness; for the treatment of problems of that kind the linearized plate theory falls short. Now, a very practical and empirically closely agreeing theory is arrived at by extending the elementary formulas so that the quadratic portions in the deflections w are retained in the changes of the coefficients of the linear element[1] ("strain" and "slippage") while, as before, the higher powers of the stretch u,v and likewise the products $s\,w$ and $z\,w$ (because s is of the same order of magnitude as w) are stricken from higher than the second order.

If γ denotes the changes of the coefficients of the line element, and $\overline{\gamma}$ its mean values over the plate thickness (strain portions), we have:

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[1]A more elaborate argumentation of this hypothesis is found in a report by E. Trefftz, entitled "On the Derivatives of Stability Criteria ... III." Intern. Mech. Kongress, Stockholm, vol. III, 1930, p. 44; s.a. Handbuch der Physik, vol. VI, p. 56.

$$\left.\begin{aligned}
\gamma_{11} &= \overline{\gamma}_{11} - 2z\,\frac{\partial^2 w}{\partial x^2},\quad \gamma_{22} = \overline{\gamma}_{22} - 2z\,\frac{\partial^2 w}{\partial y^2},\\
&\qquad\qquad \gamma_{12} = \overline{\gamma}_{12} - 2z\,\frac{\partial^2 w}{\partial x \partial y},\\
&\text{with}\\
\overline{\gamma}_{11} &= 2\,\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2,\quad \overline{\gamma}_{22} = 2\,\frac{\partial v}{\partial y} + \left(\frac{\partial w}{\partial y}\right)^2,\\
&\qquad\qquad \overline{\gamma}_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}
\end{aligned}\right\} \tag{2.1}$$

whereby the terms w_x^2, w_y^2, $w_x\, w_y$ are new compared to classical theory.

All other assumptions of the elasticity theory of small deformations may be retained unaltered. In particular, Hooke's law in original form:

$$\sigma_x = E'\left(\frac{\gamma_{11}}{2} + \nu\,\frac{\gamma_{22}}{2}\right),\quad \sigma_y = E'\left(\frac{\gamma_{22}}{2} + \nu\,\frac{\gamma_{11}}{2}\right),\quad \tau = G\,\gamma_{12}$$

$$\left(E' = \frac{E}{1 - \nu^2}\right) \tag{2.2}$$

and the terms:

$$\left.\begin{aligned}
a_3 &= \frac{1}{2}\left[\sigma_x\,\frac{\gamma_{11}}{2} + \sigma_y\,\frac{\gamma_{22}}{2} + \tau\,\gamma_{12}\right]\\
&= \frac{E'}{8}\,[(\gamma_{11}+\gamma_{22})^2 - 2\,(1-\nu)\,(\gamma_{11}\gamma_{22}-\gamma_{12}^2)],\\
&= \frac{1}{2E}\,[(\sigma_x+\sigma_y)^2 - 2\,(1+\nu)\,(\sigma_x\sigma_y-\tau^2)]
\end{aligned}\right\} \tag{2.3}$$

for the form change energy (FE, abbreviated) a_3 per unit volume retain their validity.

The FE per unit surface a_2 is obtained, for example, from equation (2.3) by integrating z from $-s/2$ to $+s/2$ as the sum of the "strain energy" $\overline{a}_2$ and the "bending energy" $\tilde{a}_2$

$$a_2 = \overline{a}_2 + \tilde{a}_2$$

with

$$\left.\begin{aligned}\bar{a}_2 &= E' \frac{s}{2} \left\{ \left[u_x + v_y + \frac{1}{2} w_x^2 + \frac{1}{2} w_y^2 \right]^2 \right. \\ &\qquad - 2\,(1-\nu) \left[\left(u_x + \frac{1}{2} w_x^2 \right) \left(v_y + \frac{1}{2} w_y^2 \right) \right. \\ &\qquad\qquad \left.\left. - \left(u_y + v_x + w_x\, w_y \right)^2 \right] \right\} \\ \tilde{a}_2 &= E' \frac{s^3}{24} \left\{ [w_{xx} + w_{yy}]^2 - 2\,(1-\nu)\, [w_{xx} w_{yy} - w_{xy}^2] \right\}\end{aligned}\right\} \qquad (2.4_1)$$

For the practical calculation it is of advantage to introduce the stresses rather than the form changes in the strain portion $\bar{a}_2$. Dividing the stress $\sigma\,(x,y,z)$ prevailing at any point within the plate in the conventional manner into strain $\bar{\sigma}$ and bending stresses $\tilde{\sigma}$ (say, in form of

$$\sigma_x = \bar{\sigma}_x - \frac{2z}{s}\,\tilde{\sigma}_x, \quad \sigma_y = \bar{\sigma}_y - \frac{2z}{s}\,\tilde{\sigma}_y, \quad \tau = \bar{\tau} - \frac{2z}{s}\,\tilde{\tau}\,)$$

the strain portion expresses itself through the stress mean values $\bar{\sigma}$ in the form of

$$\bar{a}_2 = \frac{s}{2E} \left[(\bar{\sigma}_x + \bar{\sigma}_y)^2 - 2\,(1+\nu)\,(\bar{\sigma}_x\,\bar{\sigma}_y - \bar{\tau}^2) \right] \qquad (2.4_2)$$

On the other hand, the strain stresses follow Cauchy's equilibrium equations:

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}}{\partial y} = 0, \quad \frac{\partial \bar{\tau}}{\partial x} + \frac{\partial \bar{\sigma}_y}{\partial y} = 0 \qquad (2.5)$$

and the form of these equations allows us to replace the three unknown functions $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}$ by one "stress function" Φ through:

$$\bar{\sigma}_x = \Phi_{yy}, \quad \bar{\sigma}_y = \Phi_{xx}; \quad \bar{\tau} = -\,\Phi_{xy} \qquad (2.6)$$

Hence the expression (2.4_2) for the strain portion of FE becomes:

$$\bar{a}_2 = \frac{s}{2E} \left[(\Phi_{xx} + \Phi_{yy})^2 - 2\,(1+\nu)\,(\Phi_{xx}\Phi_{yy} - \Phi_{xy}^2) \right] \qquad (2.4_3)$$

and the total form change energy A stored in a plate of length l and width b may, according to (2.4_3) and

(2.4_1) be written as:

$$A = \int_{-l/2}^{l/2} \int_{-b/2}^{b/2} \left\{ \frac{S}{2E} [(\Delta\Phi)^2 - 2 (1+\nu) (\Phi_{xx}\Phi_{yy} - \Phi_{xy}{}^2)] \right.$$
$$\left. + \frac{E' s^3}{24} [(\Delta w)^2 - 2 (1-\nu) (w_{xx}w_{yy} - w_{xy}{}^2)] \right\} dx\, dy \qquad (2.7)$$

An equation between stress function Φ and deflection w is obtained if Hooke's law (2.2) is specially rewritten for the mean values (strain portions) of the stresses and strains:

$$\left.\begin{aligned} \Phi_{yy} &= E' \left(u_x + \nu v_y + \tfrac{1}{2} (w_x{}^2 + \nu w_y{}^2)\right) \\ \Phi_{xx} &= E' \left(v_y + \nu u_x + \tfrac{1}{2} (w_y{}^2 + \nu w_x{}^2)\right) \\ \Phi_{xy} &= G (v_x + u_y + w_x w_y) \end{aligned}\right\} \qquad (2.8)$$

and the displacement u, v is eliminated from these three equations (reference 1):

$$\Delta \Delta \Phi = E (w_{xy}{}^2 - w_{xx} w_{yy}) \qquad (2.9)$$

With the aid of equations (2.6) to (2.9) the condition of stress and strain in the plate can be progressively determined (Ritz's method).[2]

III. DETERMINATION OF STRESS FUNCTION Φ AND OF DEFLECTION f AS FUNCTION OF ϵ_1 AND ϵ_2

In contradistinction to the pure energy method employed in the report quoted at the beginning, which makes explicit use of only equations (2.2) and (2.4_1) we apply

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[2] There is little hope for an exact integration of (2.9) together with the "extended plate theory" which might be added as second equation to determine Φ and w (cf. K. Marguerre: Z.f.a.M.M., vol. 16, 1936, p. 353), because both equations are non-linear in addition to being coupled (in contrast to the elementary theory).

a "mixed" method. The deflection w is again expressed by a Ritz formula (containing arbitrary values), but we first compute Φ from the differential equation (2.9), and then only proceed to the energy expression in the form of (2.7) in order to determine, by virtue of the minimal requirement of the principle of virtual displacement, the free values as functions of the independent ϵ_1 and ϵ_2.

Even though this method does not suggest itself as readily as the previously employed one, it has (and this is of particular advantage in more complicated cases - restraint, shear) the advantage of minimizing the paper work.[3]

In fact, the simplest Ritz formula[4] imaginable for w:

$$w = f \cos \frac{\pi x}{l} \cos \frac{\pi y}{b} \quad , \tag{3.1}$$

that is, assuming that this form of buckle produced at the instant of buckling, is preserved in form even beyond some distance after exceeding the buckling load (i.e., that only the free value f changes as the load increases), gives, for the right-hand side of equation (2.9):

$$E' \frac{\pi^4}{4l^2} \frac{f^2}{b^2} \left[\left(1 - \cos \frac{2\pi x}{l}\right)\left(1 - \cos \frac{2\pi y}{b}\right) - \left(1 + \cos \frac{2\pi x}{l}\right)\left(1 + \cos \frac{2\pi y}{b}\right)\right]$$

and a particular integral of the equation:

$$\Delta \Delta \Phi = - E \frac{\pi^4}{2b^2} \frac{f^2}{l^2} \left(\cos \frac{2\pi x}{l} + \cos \frac{2\pi y}{b}\right) \tag{3.2}$$

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[3]Both methods are identical in nature because the differential equations (48) for the displacements u, v are simply the equilibrium equations (2.5) written in the displacements.

[4]If $l \gg b$ this theorem must be extended to include the nodal points in transverse direction. Then l denotes the distance of two nodal lines.

can be given at once in the form of

$$\Phi_p = - E \frac{f^2}{32} \left[\left(\frac{l}{b}\right)^2 \cos \frac{2\pi x}{l} + \left(\frac{b}{l}\right)^2 \cos \frac{2\pi y}{b} \right]$$

The complete solution for Φ satisfying the limiting conditions is obtained with appropriate solutions of the correlated homogeneous equation:

$$\Delta \Delta \Phi = 0 \tag{3.3}$$

Assuming predetermined displacements, the required limiting conditions are:

$$u\left(\pm \frac{l}{2}, y\right) = \mp \epsilon_1 \frac{l}{2}, \quad v\left(x, \pm \frac{b}{2}\right) = \mp \epsilon_2 \frac{b}{2} \tag{3.4$_1$}$$

for the displacements normal to the sides;

$$v\left(\pm \frac{l}{2}, y\right) = \mp \epsilon_2 y, \quad u\left(x, \pm \frac{b}{2}\right) = \mp \epsilon_1 x \tag{3.4$_2$}$$

for the displacements tangential to the sides.

These conditions are not exactly satisfied by combination of the particular integral with a finite number of solutions of equation (3.3).

On the other hand, it has been proved that the careful compliance of the limiting conditions for the displacement tangential to the sides has no marked effect on the sought-for stress and form change condition of the plate and particularly, on the stress-strain curve. The requirement of vanishing derivation of normals instead of (3.4$_2$)

$$\frac{\partial v\left(\pm \frac{l}{2}, y\right)}{\partial x} = 0, \quad \frac{\partial u\left(x, \pm \frac{b}{2}\right)}{\partial y} = 0 \tag{3.4$_3$}$$

along the cited sides - mechanically expressed "disappearing shear"[5] - may in consequence be introduced.

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5 $$\frac{1}{G} \tau \left(\pm \frac{l}{2}, y\right) = \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]_{x=\pm l/2}$$

$$\frac{1}{G} \tau \left(x, \pm \frac{b}{2}\right) = \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]_{y=\pm b/2}$$

Both expressions disappear, in fact.

The limiting conditions (3.4_1) and (3.4_3) can by a happy chance, be complied with in very simple manner. Putting

$$-\Phi = E\,\frac{f^2}{32}\left[\left(\frac{l}{b}\right)^2 \cos\frac{2\pi\ x}{l} + \left(\frac{b}{l}\right)^2 \cos\frac{2\pi\ y}{l}\right] + \frac{1}{2}\,p_2\ x^2 + \frac{1}{2}\,p_1\ y^2 \qquad (3.5)$$

that is,

$$\left.\begin{aligned} -\bar{p}_y &= \Phi_{xx} = E\,\frac{\pi^2\ f^2}{8b^2}\cos\frac{2\pi\ x}{l} - p_2, \\ -\bar{p}_x &= \Phi_{yy} = E\,\frac{\pi^2\ f^2}{8\ l}\cos\frac{2\pi\ y}{b} - p_1 \end{aligned}\right\} \qquad (3.6)$$

equation (2.8), that is:

$$E\,u_x = \Phi_{yy} - \nu\,\Phi_{xx} - \frac{E}{2}\,w_x{}^2, \quad E\,v_y = \Phi_{xx} - \nu\,\Phi_{yy} - \frac{E}{2}\,w_y{}^2$$

and (3.4_1), that is:

$$E\,\epsilon_1 = -\frac{2}{l}\left[\int_0^{l/2}(\Phi_{yy} - \nu\,\Phi_{xx})\,dx - \frac{E}{2}\int_0^{l/2} w_x{}^2\,dx\right]$$

$$= -E\,\frac{\pi^2 f^2}{8l^2}\cos\frac{2\pi y}{b} + p_1 - \nu\,p_2 + E\,\frac{\pi^2 f^2}{8l^2}\left(1 + \cos\frac{2\pi y}{b}\right)$$

$$E\,\epsilon_2 = -\frac{2}{b}\left[\int_0^{b/2}(\Phi_{xx} - \nu\,\Phi_{yy})\,dy - \frac{E}{2}\int_0^{b/2} w_y{}^2\,dy\right]$$

$$= -E\,\frac{\pi^2 f^2}{8b^2}\cos\frac{2\pi x}{l} + p_2 - \nu\,p_1 + E\,\frac{\pi^2 f^2}{8b^2}\left(1 + \cos\frac{2\pi x}{l}\right)$$

give for the two constants p_1 and p_2 the equations:

$$\left.\begin{aligned} p_1 &= E'\left[\epsilon_1 + \nu\,\epsilon_2 - \frac{\pi^2}{8}\,f^2\left(\frac{1}{l^2} + \frac{\nu}{b^2}\right)\right] \\ p_2 &= E'\left[\epsilon_2 + \nu\,\epsilon_1 - \frac{\pi^2}{8}\,f^2\left(\frac{1}{b^2} + \frac{\nu}{l^2}\right)\right] \end{aligned}\right\} \qquad (3.7)$$

while the condition $-\Phi_{xy} = \tau = 0$ is identically fulfilled in the entire plate and consequently, also, in the sides.

The mechanical significance of constants p_1 and p_2 appears from the equations (3.6). Integration over the width and length of the plate gives:

$$\int_{-l/2}^{l/2} \Phi_{xx}\, dx = -p_2\, l, \quad \int_{-b/2}^{b/2} \Phi_{yy}\, dy = -p_1\, b$$

or

$$p_1 = \frac{1}{b} \int_{-b/2}^{b/2} \overline{p}_x\, dx, \quad p_2 = \frac{1}{l} \int_{-l/2}^{l/2} \overline{p}_y\, dy$$

In this manner the equations (3.7) give at once the sought-for relations existing between the mean values of the stress and the mean crushing ϵ_1 and ϵ_2, leaving the deflection f as the sole unknown quantity.

The determination of f follows according to the principle of virtual displacement from the premise that the potential energy (i.e., the difference between the FE and the potential of the external forces) becomes a minimum. Having assumed rigid side beams while investigating the equilibrium condition by predetermined displacements of the sides, the external forces contribute no work on the sides (fixed during the virtual displacement), and the minimum requirement for the potential energy reduces to the minimum requirement for the FE itself.

As a result, we need to formulate only (cf. equations (2.7), (3.1), and (3.5)):

$$A = \frac{s}{2E} \int [(\Delta\Phi)^2 - 2\,(1+\nu)\,(\Phi_{xx}\,\Phi_{yy} - \Phi_{xy}{}^2)]\, dx\, dy$$

$$+ \frac{E' s^3}{24} \int [(\Delta w)^2 - 2\,(1-\nu)\,(w_{xx} w_{yy} - w_{xy}{}^2)]\, dx\, dy$$

$$= \frac{sbl}{2} \left[\frac{E}{2} \left(\frac{\pi^4 f^4}{64 b^4} + \frac{\pi^4 f^4}{64 l^4}\right) + \frac{1}{E}\,(p_1 + p_2)^2 - \frac{2\,(1+\nu)}{E}\, p_1\, p_2\right]$$

$$+ \frac{E'\, s^3\, bl}{96} \left[\frac{\pi^2 f}{l^2} + \frac{\pi^2 f}{b^2}\right]^2 = A\,(p,f)$$

and to substitute the strains for the stresses by means of (3.7), giving:

$$A=A(\epsilon, f)E' s b l \left\{\frac{\epsilon_1^2}{2} + \frac{\epsilon_2^2}{2} + \nu\,\epsilon_1\epsilon_2 - \frac{\pi^2 f^2}{8 b^2}\left[\epsilon_1\left(\frac{b^2}{l^2} + \nu\right)\right.\right.$$

$$\left.+ \epsilon_2\left(1 + \nu\frac{b^2}{l^2}\right)\right] + \frac{\pi^4 f^4}{256\; b^4}\left[(3-\nu^2)\left(\frac{b^4}{l^4} + 1\right) + 4\nu\frac{b^2}{l^2}\right]$$

$$\left. + \frac{\pi^4 s^2 f^2}{96\; b^2}\left(\frac{b^2}{l^2} + 1\right)^2\right\} \qquad (3.8)$$

and $\frac{\partial A(\epsilon, f)}{\partial f} = 0$ gives for f the equation:

$$\epsilon_1\left(\frac{b^2}{l^2} + \nu\right) + \epsilon_2\left(1 + \nu\frac{b^2}{l^2}\right) = \frac{\pi^2\; s^2}{12\; b^2}\left(\frac{b^2}{l^2} + 1\right)^2$$

$$+ \frac{\pi^2\; f^2}{16\; b^2}\left[(3 - \nu^2)\left(\frac{b^4}{l^4} + 1\right) + 4\nu\frac{b^2}{l^2}\right] \qquad (3.9)$$

The three equations of (3.7) and (3.9) represent the sought-for stress-strain law above the critical load.

The result is discussed in greater detail when confined to the case of the square plate ($l = b$).

IV. STRESS-STRAIN EQUATIONS FOR THE SQUARE PLATE

The strain stresses are, according to (3.6) and (3.7):

$$\left.\begin{aligned}
\bar{p}_x &= E'\left\{\epsilon_1 + \nu\,\epsilon_2 - \frac{\pi^2 f^2}{8\; b^2}(1+\nu)\left[1 + (1-\nu)\cos\frac{2\pi y}{b}\right]\right\} \\
\bar{p}_y &= E'\left\{\epsilon_2 + \nu\,\epsilon_1 - \frac{\pi^2 f^2}{8\; b^2}(1+\nu)\left[1 + (1-\nu)\cos\frac{2\pi x}{b}\right]\right\}
\end{aligned}\right\} \qquad (4.1)$$

$$\bar{\tau} = 0 \qquad (4.2)$$

The shear stress disappears, the normal stresses are distributed at right angles to the direction of action, according to a cosine law.

For the mean values it gives with (3.9):

$$\frac{\pi^2 f^2}{8b^2} = \frac{1}{3 - \nu}\left[(\epsilon_1 + \epsilon_2) - \frac{1}{1 + \nu}\,\frac{\pi^2 s^2}{3b^2}\right] \tag{4.3}$$

$$\left.\begin{aligned} p_1 &= E'\,\frac{1 - \nu}{3 - \nu}\left\{2\epsilon_1 - (1 - \nu)\,\epsilon_2 + \frac{1}{1 - \nu}\,\frac{\pi^2 s^2}{3b^2}\right\} \\ p_2 &= E'\,\frac{1 - \nu}{3 - \nu}\left\{2\epsilon_2 - (1 - \nu)\,\epsilon_1 + \frac{1}{1 - \nu}\,\frac{\pi^2 s^2}{3b^2}\right\} \end{aligned}\right\} \tag{4.4}$$

For the maximum value of the compression $\bar{p}_x$ (occurring on the sides), it is:

$$p_R = E'\,\frac{1 - \nu}{3 - \nu}\left\{(3+\nu)\,\epsilon_1 + \nu\left[2\epsilon_2 + \frac{1}{1 - \nu}\,\frac{\pi^2 s^2}{3b^2}\right]\right\} \tag{4.5}$$

while for the minimum value (in the middle), we obtain:

$$p_M = E'\,\frac{1 - \nu}{3 - \nu}\left\{(1-\nu)\left[\epsilon_1 - 2\epsilon_2 + \frac{2 - \nu}{1 - \nu}\,\frac{\pi^2 s^2}{3b^2}\right]\right\} \tag{4.6}$$

In many cases it is of advantage to treat the stress and strain condition as function of the mean compression p_1 and p_2 exerted on the side beams (as independent variable). The solution of the system (4.4) gives:

$$\left.\begin{aligned} E\,\epsilon_1 &= 2p_1 + (1 - \nu)\,p_2 - p_o \\ E\,\epsilon_2 &= 2p_2 + (1 - \nu)\,p_1 - p_o \end{aligned}\right\} \tag{4.7}$$

where p_o is an abbreviation for

$$p_o = E'\,\frac{\pi^2 s^2}{3b}$$

then (4.3) is replaced by

$$E\,\frac{\pi^2 f^2}{8b^2} = p_1 + p_2 - p_o \tag{4.8}$$

and (4.1) reduces to the simple form of

$$\left.\begin{aligned}\overline{p}_x &= p_1 - (p_1 + p_2 - p_o)\cos\frac{2\pi y}{b}\\ \overline{p}_y &= p_2 - (p_1 + p_2 - p_o)\cos\frac{2\pi x}{b}\end{aligned}\right\} \qquad (4.9)$$

V. THE SPECIAL CASES $p_2 = 0$ AND $\epsilon_2 = 0$

The equations (4.1) to (4.9) become considerably more simple when further divided, for two very important practical cases:

<u>Freely shifting longitudinal sides</u> $p_2 = 0$	<u>Nonshifting longitudinal sides</u> $\epsilon_2 = 0$

From the second equation (4.4) and (4.7) follows:

$$\epsilon_2 = \frac{1-\nu}{2}\epsilon_1 - \frac{1}{2(1-\nu)}\frac{\pi^2 s^2}{3b^2} \qquad\Big|\qquad p_2 = -\frac{1-\nu}{2}p_1 + \frac{1}{2}p_o$$

and consequently,

$$\frac{\pi^2 f^2}{8b^2} = \frac{1}{2}\left(\epsilon_1 - \frac{1}{1-\nu^2}\frac{\pi^2 s^2}{3b^2}\right) \qquad\Big|\qquad E\frac{\pi^2 f^2}{8b^2} = \frac{1+\nu}{2}\left(p_1 - \frac{p_o}{1+\nu}\right) \qquad (5.1_1)$$

from (4.3) and (4.8).

On the other hand, (4.8) and (4.3) give at once:

$$E\frac{\pi^2 f^2}{8b^2} = (p_1 - p_o) \qquad\Big|\qquad \frac{\pi^2 f^2}{8b^2} = \frac{1}{3-\nu}\left(\epsilon_1 - \frac{1}{1+\nu}\frac{\pi^2 s^2}{3b^2}\right) \qquad (5.1_2)$$

With $f = 0$ these relations give the critical values:

$$\epsilon_1\,(f=0) = \epsilon^* = \frac{1}{1-\nu^2}\frac{\pi^2 s^2}{3b^2} \qquad\Big|\qquad \epsilon_1\,(f=0) = \epsilon^{**} = \frac{1}{1+\nu}\frac{\pi^2 s^2}{3b^2}$$

$$p_1\,(f=0) = p^* = p_o \qquad\Big|\qquad p_1\,(f=0) = p^{**} = \frac{p_o}{1+\nu}$$

$$= \frac{E}{1-\nu^2}\frac{\pi^2 s^2}{3b^2} = E\,\epsilon^* \qquad\Big|\qquad = E'\frac{1}{1+\nu}\frac{\pi^2 s^2}{3b^2} = E'\,\epsilon^{**} \qquad (5.2)$$

in accord with known results.

The second equation of (4.4) discloses that the coefficient prefix for the transverse strain (or transverse compression) changes when

$$\epsilon_1 = \left(\frac{1}{1 - \nu}\right)^2 \frac{\pi^2 s^2}{3\ b^2}$$

Referred to the critical crushing ϵ^* or ϵ^{**}, it means that (for a transverse contraction figure $\nu = 1/3$):

The transverse elongation becomes a contraction by twice exceeded buckling crushing

The transverse compression becomes a tensile stress by three times exceeded buckling crushing

With the abbreviations (5.2), the equations (4.3) and (4.8) read:

$$\frac{\pi^2 f^2}{4 b^2} = \epsilon_1 - \epsilon^* \qquad \Bigg| \qquad \frac{\pi^2 f^2}{4 b^2} = \frac{2}{3 - \nu} (\epsilon_1 - \epsilon^{**})$$

$$E\ \frac{\pi^2 f^2}{4 b^2} = 2\ (p_1 - p^*) \qquad \Bigg| \qquad E'\ \frac{\pi^2 f^2}{4 b^2} = \frac{1}{1 - \nu} (p_1 - p^{**}) \tag{5.4}$$

Eliminating f from these two pairs of equations leaves the stress-strain law in the form:

$$p_1 - p^* = \frac{1}{2}\ E\ (\epsilon_1 - \epsilon^*) \qquad \Bigg| \qquad p_1 - p^{**} = E'\ \frac{2\ (1 - \nu)}{3 - \nu}\ (\epsilon_1 - \epsilon^{**}) \tag{5.5}$$

Note that the relationship of stress and strain remains linear even above the buckling load within the scope of our approximation; the "apparent strain stiffness" is

Unrelated to ν for $\nu = \frac{1}{3}$

is exactly half as great as in the proper elastic range below the critical load.

Observing that in both cases the compressive stress

$$p_L = E\ \epsilon_1$$

prevails in the longitudinals, equation (5.5) gives for the apparent width b_m of the sheet, according to the equation

$$p_L \, b_m = p_1 \, b$$

$\frac{b_m}{b} = \frac{p^*}{p_L} + \frac{1}{2}\left(1 - \frac{p^*}{p_L}\right) = \frac{1}{2}\left(1 + \frac{p^*}{p_L}\right)$	$\frac{b_m}{b} = \frac{2}{(3-\nu)(1+\nu)}\left(1 + \frac{(1+\nu)^2}{2}\frac{p^{**}}{p_L}\right) = \frac{9}{16}\left(1 + \frac{8}{9}\frac{p^{**}}{p_L}\right) \quad (\nu = 1/3)$

(5.6)

The transverse tension set up by nonshifting sides causes a slight increase in apparent width (i.e., in the load capacity of the sheet) relative to the case of disappearing mean transverse stress. The same result obtains from a comparison of the minimum compression values produced in the centroidal axis: Substituting the side stress p_R and the critical compression p^* and p^{**} for ϵ_1 and ϵ^* (ϵ^{**}), according to (4.5) and (5.2), equation (4.6) gives for p_M the values:

$p_M = p^*$	$p_M = \frac{1-\nu}{3+\nu}\left(p_R + \frac{2(1+\nu)}{1-\nu}p^{**}\right) = 1/5 \, p_R + 4/5 \, p^{**} \quad (\nu=1/3)$

(5.7)

where it will be observed that in the first case the centroidal axis rejects every compression rise above buckling, while in the second case, because of its better support by the transverse fibers, it takes up (even though small) a part of the compression.

The result expressed by equation (5.5) - that the apparent strain stiffness in a square plate at the instant of buckling reduces by half, is quite generally valid for

$p_2 = \text{const} \neq 0$	$\epsilon_2 = \text{const} \neq 0$

The critical value itself is affected by the magnitude of

p_2 or $\epsilon_2 \lesseqgtr 0$. We find:

$$p_{kr} = p^* - p_2 \qquad\qquad \epsilon_{kr} = \epsilon^{**} - \epsilon_2$$

$$E\,\epsilon_{kr} = p^* - (1+\nu)\,p_2 \qquad\qquad p_{kr} = E'\,[\epsilon^{**} - (1-\nu)\,\epsilon_2]$$

$$= p_{kr} - \nu\,(p^* - p_{kr}) \qquad\qquad = E'\,[\epsilon_{kr} + \nu\,(\epsilon^{**} - \epsilon_{kr})] \tag{5.8}$$

and from (4.7) and (4.4) follows:

$$E\,(\epsilon_1 - \epsilon_{kr}) = [2p_1 - p_{kr} - \nu(p^* - p_{kr})] - [p_{kr} - \nu(p^* - p_{kr})] = 2\,(p_1 - p_{kr})$$

$$p_1 - p_{kr} = E'\,\frac{1-\nu}{3-\nu}\left\{\left[2\epsilon_1 + (1-\nu)\epsilon_{kr} + \frac{3\nu-\nu^2}{1-\nu}\,\epsilon^{**}\right] - \left[(3-\nu)\epsilon_{kr} - \frac{\nu\,(3-\nu)}{1-\nu}\,\epsilon^{**}\right]\right\} = E'\,\frac{2(1-\nu)}{3-\nu}\,(\epsilon_1 - \epsilon_{kr}) \tag{5.9}$$

that is, precisely the law (5.5) except for the values p_{kr}.... according to (5.8) instead of the values p^*, etc.

The linear aspect of the stress-strain curve above the critical load is, of course, a result of the limitation to the approximate formula (3.1). As a matter of fact, equation (3.1) is valid only "at the very first instant" after exceeding the buckling load. The straight lines (5.5) are the tangents to the actual stress-strain curve, which, starting from these tangents, deflects downward (fig. 2).

VI. COMPARISON WITH OTHER TEST DATA

The investigations discussed thus far differ from older reports on this subject (Schnadel, reference 2; Cox, reference 3, Yamamoto and Kondo, reference 4, Timoshenko, reference 5), in sofar as, other than hypothesis (3.1)

$$w = f \cos\frac{\pi x}{b} \cos\frac{\pi y}{b} \quad \text{(square plate)} \tag{6.1}$$

concerning the buckling form, no other arbitrary assumptions are introduced.

Instead of determining the horizontal displacements u and v explicitly - or by introducing the stress function Φ directly - the first three authors cited, introduce the assumption $\bar{\tau} = 0$ or in other words, $\frac{\partial\bar{\sigma}_x}{\partial x} = 0$, $\frac{\partial\bar{\sigma}_y}{\partial y} = 0$. By virtue of this assumption,[6] $\bar{\sigma}_x$ may be written in the form: $\bar{\sigma}_x = \frac{1}{b}\int_{-b/2}^{b/2} \bar{\sigma}_x \, dx$ and the extended Hooke's law (2.8):

$$\frac{1}{b}\int_{-b/2}^{b/2} \bar{\sigma}_x \, dx = \frac{E'}{2b}\int_{-b/2}^{b/2} (\bar{\gamma}_{11} + \nu\,\bar{\gamma}_{22})\, dx$$

$$= \frac{E'}{b}\left[\int_{-b/2}^{b/2}\left(u_x + \frac{1}{2}\,w_x^2\right) dx + \nu\int_{-b/2}^{b/2}\left(v_y + \frac{1}{2}\,w_y^2\right) dx\right] \qquad (6.2)$$

changes in the special case of $\nu = 0$ to

$$\left.\begin{aligned}\bar{\sigma}_x &= \frac{E}{b}\left[(u(b/2) - u(-b/2)\right] + \frac{E}{2b}\int_{-b/2}^{b/2} w_x^2\, dx \\ &= E\left[-\epsilon_1 + \frac{1}{2b}\int_{-b/2}^{b/2} w_x^2\, dx\right] \\ &\text{and correspondingly:} \\ \bar{\sigma}_y &= E\left[-\epsilon_2 + \frac{1}{2b}\int_{-b/2}^{b/2} w_y^2\, dy\right]\end{aligned}\right\} \qquad (6.3)$$

---

[6]Admittedly, Schnadel adduces this relation by the stress-function method but the inference is right only for each single term of the formula $w = \sum_{ik} f_i \cos\frac{i\pi x}{b}\cos\frac{k\pi y}{b}$. As soon as several summands are admitted simultaneously, the law of superposition naturally ceases to hold and, in fact, the mixed terms $f_i f_k$ lead to a nondisappearing shear stress whose omission, to be sure, renders the calculation much easier.

Then the strain portion $\bar{A}$ of FE becomes in the case of disappearing transverse contraction:

$$\bar{A} = \frac{s}{2E}\iint (\sigma_x{}^2+\sigma_y{}^2)\ dx\ dy = \frac{bs}{2E}\left[\int \sigma_x{}^2\ dy + \int \sigma_y{}^2\ dx\right] =$$

$$= \frac{Es}{2}\left[\int\left(\epsilon_1 - \frac{1}{2b}\int w_x{}^2\ dx\right)^2 dy + \int\left(\epsilon_2 - \frac{1}{2b}\int w_y{}^2\ dy\right)^2 dx\right] \quad (6.4)$$

which means that this portion (by given Ritz formula for w) can be determined in very simple fashion.

The energy of bending is given unchanged according to (2.4) or (2.7).

Cox and Yamamoto-Kondo, after him, carry this simplification one step farther by writing $\bar{\sigma}_y(x,y) = 0$ for the whole plate; that is, neglect even the work of strain performed by the transverse stresses. Schnadel, on the contrary, uses (6.4) as basis[7]. Allowance for the transverse contraction is in none of the formulas possible.

Cox's assumption $\bar{\sigma}_y\ (x,y) = 0$ contains two statements: that the transverse stress within is invariable $\left(\frac{\partial\bar{\sigma}_y(xy)}{\partial y} = 0\right)$ and that it disappears at the side $(\sigma_y\ (x, \pm\ b/2) = 0)$. In the practical, most important case of strong - i.e., especially undeflectable longitudinals - the limiting condition is certainly not complied with; and so for this reason alone a direct comparison of Cox's results with ours is impossible. Granted even that we visualize the limiting condition $\bar{\sigma}_y = \bar{\tau} = 0$ to have been realized by some appropriate test arrangement, the first statement regarding the strain condition within, still remains fundamentally inadmissible. Because, in order to apply Ritz's method in a mathematically unobjectionable manner (that is, for example, preserve the known dictum that the true load capacity must lie below that computed by approximate formula), it is not permitted to

[7] Since Schnadel proceeds from a somewhat different concept of the buckling process (division in internal and external energy), the true facts of the case are not apparent at once.

mix geometrical statements (concerning w) with mechanical ones (about $\overline{\tau}$). Unless the mechanical theorem $\overline{\tau} = 0$ happens to be fulfilled by itself, the FE must be reduced by this assumption: concerning the direction of the discrepancy from the true value (let alone an estimation of the errors), nothing more can be said. This drawback is felt most when several approximate solutions are to be compared; because then it is impossible to deduce that the formula giving the lowest load capacity must be the better one.

Mathematically, the omission of a part of the FE fares no better. So, for instance, Cox obtains for the stress-strain law above the buckling limit the expression

$$p_1 - p^* = \frac{E}{3} (\epsilon_1 - \epsilon^*) \tag{6.5}$$

i.e., a drop in (apparent) strain stiffness to one-third. But, if we compute the case of vanishing side stresses $\overline{\sigma}_y$ and $\overline{\tau}$ "exact" conformable to the method cited at the beginning of this report, we find:

$$p_1 - p^* = k\,E (\epsilon_1 - \epsilon^*) \tag{6.6}$$

where the coefficient k ranges between 0.41 ($\nu = 0$) and 0.34 ($\nu = \frac{1}{2}$) depending on the amount of transverse contraction. For $\nu = 0.3$ κ has the value 0.38; that is, below 0.50 (see equation (5.5)), but still noticeably above 0.33. Consequently, the omission of the share of the FE originating from the transverse and the shearing stress is not without some effect.

Cox's experimental results are in contradiction with his formula (6.5), but confirm our formula (5.5) very satisfactorily.

From Schnadel's data the results (for $p_2 = 0$) given in the preceding sections can be deduced in complete agreement, because the shear stress actually does disappear within the validity range of $w = f \cos \frac{\pi x}{b} \cos \frac{\pi y}{b}$, and the effect of the transverse contraction cancels out in the case of displaceable longitudinal sides.

Timoshenko elects to proceed from the exactly valid expression (2.4_1) rather than (6.4). But his calculation differs from the one given here by the introduction of approximate assumptions, each containing a free value for the displacements u and v (independent of the w formula). The results reveal the strange fact that the stress maximum does not occur on the side but on the inside at a distance depending on the degree of loading, so that the stresses actually decrease even toward the side. Apart from that, there are (as in Yamamoto's case) tensile stresses in the center of the bay under greatly exceeded buckling load. Instead of the formulas (5.5), through which the decrease in apparent strain stiffness above ϵ^* versus elastic stiffness, finds expression, we find:

$$p_1 - p^* = 0.640\ E(\epsilon_1 - \epsilon^*), \quad \Big| \quad p_1 - p^{**} = 0.624\ E'(\epsilon_1 - \epsilon^{**}) \tag{6.7}$$

at $\nu = 0.3$, according to Timoshenko's calculation. It is seen that, according to these formulas, the load capacity is rated too favorable, as it should be; because Timoshenko uses the energy term without omission while employing a Ritz formula, which does not express the actually occurring conditions as well as the one used in the present report.

By an extension of (6.1) the cited authors seek to account for the fact that the buckles must become deformed if the buckling load has been exceeded to a comparatively appreciable extent when, as a matter of fact the bulge form (6.1) is derived from the energy balance (between bending and strain energy) at the instant of buckling, and it is evident that this balance must change as the compressive forces move toward the sides. The first three authors quoted are unanimous in assuming that the profile form of the bulge must remain unchanged $(w = \cos \frac{\pi x}{b}\ \psi(y))$ in the compressive (x) direction, but that transversely to it, a flattening takes place in the middle, and a corresponding buckling near the edges. According to that assumption, the longitudinal fibers subjected to higher normal compression are curved much more, and therefore, are better able to avoid the compression, and the increase in bending energy is less than the thus resulting decrease in strain energy. Cox particularly assumes the profile form to be built up from two sine arcs at the edges and a straight piece in the middle, the length of which follows from the minimum requirement. Without attacking the other assumptions by Cox $(\bar{\sigma}_y = \bar{\tau} = 0)$, Yamamoto and Kondo turn against the arbitrariness of this assumption regarding the buckling form, which, in fact, due to the jump in the curvature, is not at all satisfactory. And that is the reason for their "exact" computation of the form of buckle by means of a nonlinear differential equation. As remarkable as their ensuing mathematical problem undoubtedly is, the labor involved is not justified in this case because: first, the solution of the problem stands and falls with the premises $\bar{\sigma}_y (x,y) = \bar{\tau} (x,y) = 0$; that is, its importance from the practical point of view is subordinate; second, this method is not worth while even in this particular case because, with the help of the continuous formula in the curvatures (see report cited at the beginning):

$$w = \cos \frac{\pi x}{b} \left(f_1 \cos \frac{\pi y}{b} - f_3 \cos \frac{3\pi y}{b}\right) \tag{6.8}$$

The two parameters f_1 and f_3 give with the energy method the same buckle pattern and the same stress-strain law with slide-rule accuracy as Yamamoto and Kondo achieved with their exact method.

Schnadel also employed this same formula (6.8). His results differ very little from those obtained without the assumption $\bar{\tau} = 0$. The difference in buckle form (6.8), and so from the stress-strain law (5.5) is, moreover, slight in the case of $\bar{\sigma}_y \neq 0$ if only one deformation of the profile $x = \text{constant}$ is taken into consideration. (This remark was also made by Timoshenko.)

The relatively good agreement of Cox's formula for the apparent width under 100 times exceeded buckling crushing, is merely accidental. Because the load capacity is considerably underestimated at the very beginning on account of the disregarded internal transverse and shearing stresses, and that necessarily is righted again under sufficiently exceeded load, because the buckling form, consistently diverging more and more from the actual form, mistakingly presents a too high load capacity.

VI. EXTENSION OF FORMULA (6.1)

The Apparent Width Under Considerably Exceeded Buckling Load

The aforementioned displacement of the pressure distribution under considerably exceeded buckling load (as often encountered in practical airplane design) calls for a much greater divergence from the buckle form (6.1) than afforded in (6.8). For it is a fact that under ten times exceeded buckling load, for example, the two strips adjacent to the edges carry nearly the full load, while the middle of the plate takes up little more than the buckling stress itself. The conception suggested by the apparent width, that the plate falls approximately into three zones: the two edge strips of width b_m (the apparent width), and the unloaded middle part are indicative of the kind of strain to be expected. Intermediate buckles must form near the edges[8] since the edge strips, exactly like the whole plate (at start of phenomenon), have the tendency to split more into square panels rather than into the very long rectangles, so that we may put

$$w = f_1 \cos\frac{\pi x}{b}\cos\frac{\pi y}{b} - f_3 \cos\frac{3\pi x}{b}\left(\cos\frac{\pi y}{b} - \eta\cos\frac{3\pi y}{b}\right) \tag{7.1}$$

with the parameters f_1, f_3, and η. For $\eta = 1$, for example, this formula would have the centroidal axis $y = 0$ retain its cosine form unchanged, while toward the edge, the middle of each panel $(x = 0)$ develops appreciable counterbuckles under sufficiently great f_3. With proper choice of $\eta < 1$ the exact location of these buckles can be more accurately determined and a certain amount of flattening obtained even in the middle.

The calculation itself may be made by either one of two methods: the "exact" method, or the approximate method evolved on the formulas (6.3) and (6.4).

The "exact" method being, as is quickly proved, extremely tedious, we give hereinafter, only the approximate method and then compare the results.

It is:

$$w=f_1 \cos\frac{\lambda x}{2}\cos\frac{\lambda y}{2} - f_3 \cos 3\frac{\lambda x}{2}\left(\cos\frac{\lambda x}{2} - \eta\cos\frac{3\lambda y}{2}\right), \quad \left(\lambda = \frac{2\pi}{b}\right)$$

$$\left(\frac{2}{\lambda}\right)^2 w_x^2 = \frac{1}{4} f_1^2 (1-\cos\lambda x)(1+\cos\lambda y) - \frac{9}{4} f_3^2 (1-\cos 3\lambda x)$$

$$[1 + \cos\lambda y - 2\eta(\cos\lambda y+\cos 2\lambda y)+\eta^2(1+\cos 3\lambda y)],$$

---

[8]The phenomenon of intermediate buckles above ten times exceeded buckling load, was experimentally observed by Lahde (Luftfahrtforschung, vol. 13, 1936, p. 214).

$$- \frac{3}{2} f_1 f_3 (\cos \lambda\ x - \cos 2\lambda\ x) \ [(1+\cos \lambda\ x) - \eta(\cos \lambda\ y + \cos 2\lambda\ y)],$$

$$\left(\frac{2}{\lambda}\right)^2 w_y^2 = \frac{1}{4} f_1^2 \ (1+\cos \lambda\ x) \ (1-\cos \lambda\ y) + \frac{1}{4} f_3 (1+\cos 3\lambda\ x)$$

$$[1 - \cos \lambda\ y - 6\eta(\cos \lambda\ y - \cos 2\lambda\ y) + 9\eta^2 \ (1-\cos 3\lambda\ y)]$$

$$- \frac{1}{2} f_1 f_3 (\cos \lambda\ y + \cos 2\lambda\ x) [(1-\cos \lambda\ y) - 3\eta(\cos \lambda\ y - \cos 2\lambda\ )],$$

$$C \equiv \frac{1}{2b} \int_0^b w_x^2 \ dx = \frac{\pi^2}{8b^2} \Big\{ f_1^2 \ (1 + \cos \lambda\ y) + 9f_3^2 \ [(1 + \cos \lambda\ y) - 2\eta \ (\cos \lambda\ y + \cos 2\lambda\ y) + \eta^2 \ (1 + \cos 3\lambda\ y)] \Big\}$$

$$D \equiv \frac{1}{2b} \int_0^b w_y^2 \ dy = \frac{\pi^2}{8b^2} \Big\{ f_1^2 \ (1 + \cos \lambda\ x) + f_3^2 \ [(1 + \cos 3\lambda\ x) \ (1 + 9\eta^2)] - 2f_1 \ f_3 \ (\cos \lambda\ x + \cos 2\lambda\ x) \Big\}$$

$$\frac{1}{b} \int_0^b C^2 \ dy = \frac{\pi^4}{64\ b^4} \Big\{ \frac{3}{2} f_1^4 + 81\ f_3^4 : \left[\frac{3}{2} (1 + \eta^4) + 6\eta^2 - 2\eta\right] + 18\ f_1^2 \ f_3^2 \left[1 + \eta^2 + \frac{1}{2} (1 - 2\eta)\right] \Big\}$$

$$\frac{1}{b} \int_0^b D^2 \ dx = \frac{\pi^4}{64\ b^4} \Big\{ \frac{3}{2} f_1^4 + \frac{3}{2} f_3^4 \ (1 + 9\eta^2)^2 + f_1^2 f_3^2 \ [4 + 2(1 + 9\eta^2)] - 2f_1^3 \ f_3 \Big\}$$

Then (6.4) gives the strain portion $\overline{A}$ of the FE at:

$$\bar{A} = G\ s\ b^2 \left\{\epsilon_1^2 - \epsilon_1 \frac{\pi}{4b^2} [f_1^2 + 9\ (1 + \eta^2)\ f_3^2] \right.$$

$$+ \epsilon_2^2 + \epsilon_2 \frac{\pi^2}{4b^2} [f_1^2 + (1 + 9\eta^2)\ f_3^2]$$

$$+ \frac{\pi^4}{64\ b^4} \left[3f_1^4 - 2f_1^3\ f_3 + 6f_1^2\ f_3^2 \left(\frac{11}{2} - 3\eta + 6\eta^2\right)\right.$$

$$\left. + f_3^4\ (123 - 162\eta + 513\eta^2 + 243\ \eta^4)\right] \tag{7.2$_3$}$$

and for the bending portion $\tilde{A}$, according to (2.7)

$$\tilde{A} = \frac{G\ s^3}{12} [4f_1^2 + f_3^2\ (100 + 324\ \eta^2)]$$

$$= \frac{G\ s^3}{3} [f_1^2 + f_3^2\ (25 + 81\ \eta^2)] \tag{7.2$_4$}$$

In view of the high degree of the ensuing equations, it is not advisable to attempt the determination of f_1, f_3, and η in relation to ϵ_1 by means of the three equations:

$$\frac{\partial A}{\partial f_1} = 0, \quad \frac{\partial A}{\partial f_3} = 0, \quad \frac{\partial A}{\partial \eta} = 0 \tag{7.3$_1$}$$

and one of the conditions of section V for ϵ_2 by the usual method, but rather to abandon the condition $\frac{\partial A}{\partial \eta} = 0$ for η, the most unessential of the parameters, and to arrive by trial and error at a value of η yielding the lowest possible load capacity. The differentiation, according to f_1 and f_3 gives:

$$\left.\begin{aligned}
&- \epsilon_1 - \epsilon_2 + \frac{\pi^2}{32\ b^2} [12\ f_1^2 + 6\ f_3^2\ (11-6\eta+12\eta^2) \\
&\qquad\qquad - 6f_1\ f_3] + \epsilon^* = 0 \\
&- 9\ (1 + \eta^2)\ \epsilon_1 - (1+9\eta^2)\ \epsilon_2 \\
&\qquad + \frac{\pi^2}{32\ b^2} \left[6f_1^2\ (11-6\eta+12\eta^2) - 2\ \frac{f_1^3}{f_3}\right. \\
&\qquad \left. + 4f_3^2\ (123 - 162\eta + 513\eta^2 + 243\eta^4)\right] \\
&\qquad\qquad + (25 + 81\eta^2)\ \epsilon^* = 0
\end{aligned}\right\} \tag{7.3$_2$}$$

It also follows from $p_2 = 0$ (the only case considered here), according to (6.3) and (7.2₁):

$$\epsilon_2 = \frac{\pi^2}{32\ b^2}\left[4f_1^2 + 4\ (1 + 9\eta^2)\ f_3^2\right] \tag{7.3_3}$$

that is:

$$\left.\begin{aligned} \epsilon_1 &= \epsilon^* + \frac{\pi^2}{16\ b^2}\left[4f_1^2 - 3f_1\ f_3 + f_3^2\ (31-18\eta+18\eta^2)\right] \\ 9\ (1 + \eta^2)\ \epsilon_1 &= (25+81\eta^2)\ \epsilon^* \\ &\quad + \frac{\pi^2}{16\ b^2}\left[f_1^2\ (31-18\eta+18\eta^2) - f_1^3/f_3 \right. \\ &\quad \left. + f_3^2\ (244 - 324\eta + 990\eta^2 + 324\eta^4)\right] \end{aligned}\right\} \tag{7.3_4}$$

The elimination of the parameter f_1 from both equations leaves:

$$\frac{\epsilon_1 - \epsilon^*}{\epsilon_1 - \dfrac{25 + 81\ \eta^2}{9 + 9\ \eta^2} - \epsilon^*} = 9\ (1 + \eta^2)$$

$$\frac{4 - 3\zeta + (31 - 18\eta + 18\eta^2)\ \zeta^2}{(31-18\eta+18\eta^2) - 1/\zeta + (244 - 324\eta + 990\eta^2 + 324\eta^4)\ \zeta^2}$$

$$(\zeta = f_3/f_1) \tag{7.4_1}$$

with which ϵ_1/ϵ^* is readily computed for every η as function of $\zeta = \frac{f_3}{f_1}$, and inversely $\zeta\left(\frac{\epsilon_1}{\epsilon^*}\right)$.

Furthermore, p_1 is given according to (6.3) and (7.2) through the equation

$$p_1 = -\frac{1}{b}\int_{-b/2}^{b/2} \bar{\sigma}_x\ dy = \epsilon_1 - \frac{\pi^2}{8b^2}\left[f_1^2 + 9\ (1+\eta^2)\ f_3^2\right] \tag{7.4_2}$$

Inserting ϵ_1 according to (7.3₄) gives:

$$p_1 = p^* + E\frac{\pi^2}{16b^2}\left[2f_1^2 - 3f_1 f_3 + (13-18\eta) f_3^2\right] \qquad (7.4_3)$$

that is,

$$\frac{p_1 - p^*}{\epsilon_1 - \epsilon^*} = \frac{E}{2}\,\frac{4 - 6\zeta + (26 - 36\eta)\zeta^2}{4 - 3\zeta + (31 - 18\eta + 18\eta^2)\zeta^2} \qquad (7.4_4)$$

This formula shows conclusively that the apparent stiffness decreases at $\zeta > 0$ (i.e., by increasing strain of the buckles). Analyzing $p_1(\epsilon_1)$ as follows, from (7.4_1) and (7.4_4) for different values of η, it is seen that with $\eta = \frac{1}{2}$ the most logical course of the stress-strain curve is obtained. Then (7.4_4) and (7.4_1) read:

$$\left.\begin{aligned} \frac{p_1 - p^*}{\epsilon_1 - \epsilon^*} &= \frac{E}{2}\,\frac{4 - 6\zeta + 8\zeta^2}{4 - 3\zeta + 26.5\,\zeta^2} = \frac{E}{2}\,\varphi(\zeta) \\ \frac{\epsilon_1 - \epsilon^*}{\epsilon_1 - 4.02\,\epsilon^*} &= 11.25\,\frac{4 - 3\zeta + 26.5\,\zeta^2}{26.5 - 1/\zeta + 350\,\zeta^2} \quad (\eta = \tfrac{1}{2}) \end{aligned}\right\} \qquad (7.5)$$

(This equation (7.5) represents the sought-for stress-strain law.)

Based upon the exact method, i.e., with consideration of the part of the FE originating with the shear stresses, we obtain (likewise for $\eta = \frac{1}{2}$) in place of (7.5) the relations:[9]

$$\left.\begin{aligned} \frac{p_1 - p^*}{\epsilon_1 - \epsilon^*} &= \frac{E}{2}\,\frac{4 - 6\zeta + 18.6\,\zeta^2}{4 - 3\zeta + 31.8\,\zeta^2} = \frac{E}{2}\,\varphi(\zeta) \\ \frac{\epsilon_1 - \epsilon^*}{\epsilon_1 - 4.02\,\epsilon^*} &= 11.25\,\frac{4 - 3\zeta + 31.8\,\zeta^2}{31.8 - 1/\zeta + 350\,\zeta^2} \end{aligned}\right\} \qquad (7.6)$$

It will be observed that the majority of the terms in these equations (7.6) are in agreement.[10] The discrepan-

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[9]The difference in both theories touches only the factor of the (mixed) term $f_1^2 f_3^2$ in (7.2_3) for the FE, since the shear stresses do not contribute to the other terms (a useful check for the "exact" method).

[10]The transverse contraction precisely cancels out in this considered case $p_2 = 0$.

cies concern only terms which make themselves felt when $\zeta = 0$ (that is, for $\epsilon_1 \gg \epsilon^*$) is considerably departed from. They act in the same direction in both equations: there is for equal ζ value, according to (7.6) a greater ϵ_1 than according to (7.5) and a greater reduction factor $\varphi(\zeta)$. As it should be, the discrepancy from (5.5) is less with the exact method than with the approximation law (6.4).

In figure 2, which gives all three curves, the curves for the ϵ_1 values diverge not only from their tangents but also from each other more and more. The result of the approximation, as expedient as the abbreviated calculation may be for a first approach, must be received with caution; its better agreement with the Lahde-Wagner test points under considerably exceeded buckling load (fig. 3) is, of course, accidental. Cox's theoretical and experimental points are also shown in figure 2, and show the good agreement of his experimental points with our own results.

Although the equations (7.6) contain the essentials regarding the behavior of the plate after buckling, it may be of general interest to give the exact formula for the apparent width. Expressing the apparent width with

$$b_m \, p_L = p_1 \, b \tag{7.7$_1$}$$

gives with $p_L = E \, \epsilon_1$ (= stiffener stress) as extension of (5.8):

$$\frac{b_m}{b} = \frac{\epsilon^*}{\epsilon_1} + \frac{1}{2} \varphi(\zeta) \left(1 \;\; \frac{\epsilon^*}{\epsilon_1}\right) \tag{7.7$_2$}$$

One may attempt to approximate this curve by a simple analytical expression. In support of von Kármán's approximate formula (reference 6), it suggests the use of a formula of the form of

$$\frac{b_m}{b} = A \sqrt{\frac{\epsilon^*}{\epsilon_1}} + B$$

In fact, a satisfactory approximation is obtained with $A = 0.81$ and $B = 0.19$ within the range of $\epsilon^* \leqslant \epsilon_1 \leqslant 60 \, \epsilon^*$ or, in other words, with the formula

$$b_m = 1.54 \sqrt{\frac{E}{p_L}} \, s + 0.19 \, b \tag{7.8}$$

Within a small intermediate range the values obtained by the formula are too small.

Another "empirical" approximate formula reads:

$$\frac{b_m}{b} = \sqrt[3]{\frac{\epsilon^*}{\epsilon_1}} \tag{7.9}$$

This formula, while giving values which are a little too low for $\epsilon_1 > 20\ \epsilon^*$ relative to (7.6) reproduces, however, the typical behavior (inclusive of $\epsilon_1 = \epsilon^*$) very well, and its marked simplicity recommends it.

The Lahde-Wagner test points in figure 3 are merely by way of reference, since they pertain to the case of nonshifting sides and perfect fixity. To become "comparable" they are "converted"; that is, the course of the points with ϵ_1/ϵ^* is taken over and only the critical value ϵ_{kr} (which, naturally, is higher for fixation), is identified with our reference point ϵ^*. On the assumption that this simple conversion method is permissible, the agreement, particularly with the formula (7.9) is very good.

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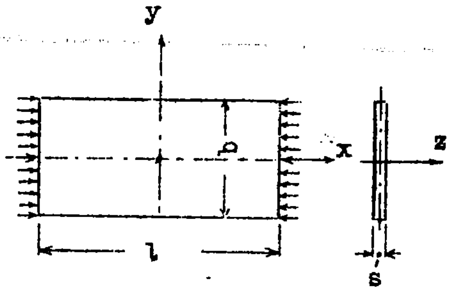
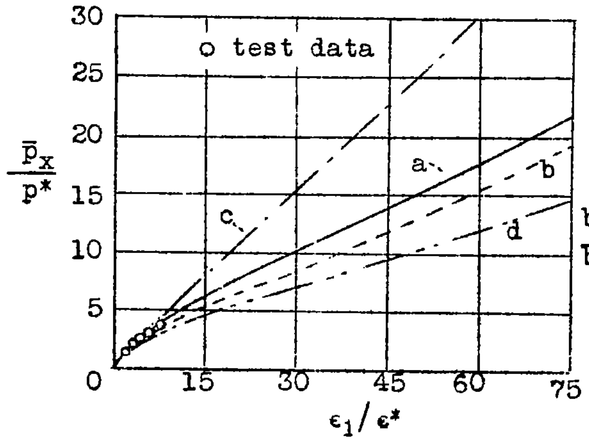
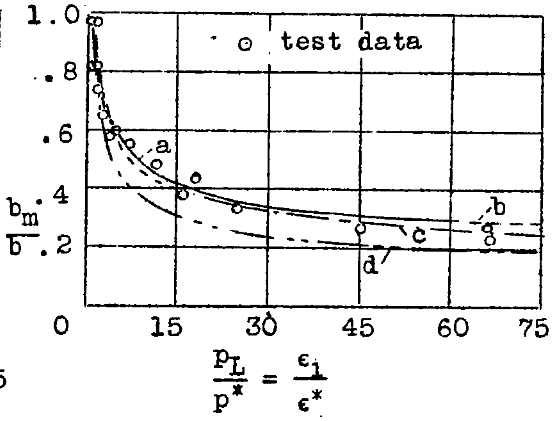


Figure 1.- Rectangular plate under uniform compression on two sides.



a. Theoretical curve(eq. 7.6)
b. " " (eq. 7.5) approximate method
c. Theoretical curve(eq. 5.6) (tangent to a and b)
d. Cox's theoretical curve

Figure 2.- Stress-strain curves according to different theories for range $\epsilon_1/\epsilon^* < 75 (p_1 = 0)$



a. Theoretical curve(eq. 7.6)
b. Approximation for a)

$$\frac{b_m}{b} = 0.81\sqrt{\frac{p^*}{p_L}} + 0.19$$

c. Approximation for a)

$$\frac{b_m}{b} = \sqrt[3]{\frac{p^*}{p_L}}$$

d. Cox's theoretical curve.

Figure 3.- The apparent width after exceeding the buckling load $p^* = E \cdot \epsilon^*$ --theoretical and experimental.